Northern Beaches Secondary College

Manly Campus

2024 **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1. Consider the two statements:

P: $\forall x \in \mathbb{R}\,,\ \exists y \in \mathbb{R}\,,\ \ y^3 = x$
Q: $\exists y \in \mathbb{R}\,,\ \forall x \in \mathbb{R}\,,\ y^3 = x$

Which statement best represents the truth of each of P and Q?

A. P is true and Q is true. B. P is true and Q is false.

C. P is false and Q is true. D. P is false and Q is false.

2. Let $\underset{\sim}{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underset{\sim}{\mathbf{b}} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

Which of the following is the value of $\underset{\sim}{\mathbf{a}} \cdot (\underset{\sim}{\mathbf{a}} - 3\underset{\sim}{\mathbf{b}})$?

A. -7 B. 0 C. 3 D. 6

3. Let $z = a + ib$, where $0 < a < b$.

Which of the following must be the case for the complex number z^4 ?

A. $Re(z^4) < 0$ B. $Re(z^4) > 0$ C. $Im(z^4) < 0$ D. $Im(z^4) > 0$

4. The velocity of an object moving in simple harmonic motion is given by $v^2 = (40 + 32x - 4x^2)$ m/s, where x is the object's displacement.

What is the period of the motion?

A. $\frac{\pi}{4}$ B. $\frac{\pi}{2}$ C. π D. 2π

5. For how many integer values of n is $n^4 + (n+i)^4$ an integer?

A. 1 B. 2 C. 3 D. 4

6. $f(x)$ and $g(x)$ are both functions with domain $(-\infty, \infty)$.

Consider the statement: "If $f(x)$ and $g(x)$ are both odd, then $f(g(x))$ is odd."

Which of the following is correct?

A. The contrapositive statement is false and the converse statement is false.

B. The contrapositive statement is true and the converse statement is true.

C. The contrapositive statement is false and the converse statement is true.

D. The contrapositive statement is true and converse statement is false.

7. Consider two spheres given by the equations $\left|\underset{\sim}{\mathbf{r}}_1\right| = 6$ and $\left|\underset{\sim}{\mathbf{r}}_2 - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}\right| = 3$.

What is the centre of the circle of intersection?

A. $(2, 0, 0)$

B. $\left(\frac{25}{8}, 0, 0\right)$

C. $\left(\frac{33}{8}, 0, 0\right)$

D. $\left(\frac{43}{8}, 0, 0\right)$

8. A complex number ω satisfies $\omega^2 + \dfrac{1}{1 + \omega^2} = \omega$.

Which of the following statements is correct?

A. $\omega^{2024} = \omega$

B. $\omega^{2024} = \omega^2$

C. $\omega^{2024} = \omega^3$

D. $\omega^{2024} = \omega^4$

9. Consider the complex numbers z of the form $z = \cos^2\theta + i\sin^2\theta$ where $0 \le \theta \le \frac{\pi}{2}$.

What is the range of the possible values of $|z|$?

A. $0 \le |z| \le 1$

B. $\frac{1}{\sqrt{2}} \le |z| \le 1$

C. $\frac{1}{2} \le |z| \le 1$

D. $\frac{1}{4} \le |z| \le 1$

10. Which of the following integrals is greater than zero?

A. $\int_{-1}^{1} \frac{x^3}{1 + x^2}\ dx$

B. $\int_{-1}^{1} x \sec x\ dx$

C. $\int_{-1}^{1} \tan^{-1}(x^2)\ dx$

D. $\int_{-1}^{1} (e^{-x^2} - 1)\ dx$

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section.

Answer each question in a separate writing booklet. Extra writing booklets are available.

For questions is Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new Writing Booklet.

a) The position vectors of two points A and B are given by

$$\overrightarrow{OA} = 2\underset{\sim}{\mathbf{i}} + 4\underset{\sim}{\mathbf{j}} - 3\underset{\sim}{\mathbf{k}} \text{ and } \overrightarrow{OB} = -\underset{\sim}{\mathbf{i}} + \underset{\sim}{\mathbf{j}} + 2\underset{\sim}{\mathbf{k}}$$

(i) Determine the exact distance between A and B. **1**

(ii) Show that there is no real value of m such that $\overrightarrow{OC} = m\underset{\sim}{\mathbf{i}} + 2\underset{\sim}{\mathbf{j}} - m^2\underset{\sim}{\mathbf{k}}$ is perpendicular to $\overrightarrow{OA}$. **2**

b) Two complex numbers are given by:

$$z_1 = 3 - i \text{ and } z_2 = 1 - 2i .$$

Determine the possible values of the real constant k if

$$\left| \frac{z_1}{z_2} + k \right| = \sqrt{k+2}$$ **3**

c) Determine the complex solutions to the equation $z^2 - (1 - 2i)z = 7 + i$ **3**

d) Consider the polynomial given by $P(z) = z^4 - 2z^3 + az^2 + bz + 50$ where $a, b \in \mathbb{R}$ and $P(2 - i) = 0$.

Fully factorise $P(z)$ such that all factors have real coefficients. **3**

e) Evaluate, in simplest exact form,

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{5 + 3\sin x - 4\cos x}\, dx$$ **3**

Question 12 (15 marks) Start a new Writing Booklet.

a) Find the fourth roots of $z = 2 + 2\sqrt{3}\, i$. **2**

b)

(i) Find A, B and C such that

$$\frac{3}{(x+1)(x^2+5)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+5}$$ **2**

(ii) Hence or otherwise, find

$$\int \frac{3}{(x+1)(x^2+5)}dx$$ **2**

c) Given that the vectors $\underset{\sim}{\mathbf{u}}$ and $\underset{\sim}{\mathbf{v}}$ satisfy $\underset{\sim}{\mathbf{u}} + \underset{\sim}{\mathbf{v}} = 17\underset{\sim}{\mathbf{i}} - \underset{\sim}{\mathbf{j}} + 2\underset{\sim}{\mathbf{k}}$ and $\underset{\sim}{\mathbf{u}} - \underset{\sim}{\mathbf{v}} = \underset{\sim}{\mathbf{i}} + 9\underset{\sim}{\mathbf{j}} - 4\underset{\sim}{\mathbf{k}}$ find the acute angle between $\underset{\sim}{\mathbf{u}}$ and $\underset{\sim}{\mathbf{v}}$ **3**

d) Prove by contradiction that there are no integers a and b such that $a^2 = 4b - 2$ **3**

e) Let a, b and c be positive integers.

Prove the following statement by contraposition:

"If $a^3 + b^3 + c^3$ is divisible by 9, then at least one of a, b and c is divisible by 3." **3**

Question 13 (15 marks) Start a new Writing Booklet.

a) Prove by mathematical induction that $7^n + 13^n + 19^n$ is divisible by 13 for all positive odd integers n. **4**

b) Find:

$$\int_1^9 \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)}\,dx$$ **3**

c)

(i) Prove that if $z = \overline{z}$ then z is real **1**

(ii) The complex numbers z and w are such that $|z| = |w| = 1$

Prove that $\dfrac{z+w}{1+zw}$ is real **3**

d) Sketch on a single Argand diagram, the region $\Re$ such that both of the following conditions are satisfied:

$$|z - i| < |z - 2 + i|$$

and

$$\frac{\pi}{4} \leq \arg(iz+1) \leq \frac{\pi}{2}$$ **4**

Question 14 (15 marks) Start a new Writing Booklet.

a) Consider the lines

$$l_1 : x = y = z$$

$$l_2 : \underset{\sim}{\mathbf{r}} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \text{ where } t \text{ is a parameter}$$

(i) Show that l_1 and l_2 are non-parallel, non-intersecting lines. **3**

(ii) Determine the angle between l_1 and l_2. **2**

b) A particle is moving in a straight line with a displacement function given by $x = 1 + \sqrt{2}\cos 2t + \sqrt{2}\sin 2t$

(i) Show that the particle's motion is simple harmonic and state the centre and period of the motion. **3**

(ii) Find the amplitude of the motion. **2**

c) Prove that for every positive real number x, there is a real number y such that $y(y+1) = x$. **2**

d) A particle's movement is represented by the vector $\underset{\sim}{\mathbf{r}}(t) = \begin{pmatrix} t + t^{-1} \\ t^3 + t^{-3} \end{pmatrix}$ where $t \in \mathbb{R}^+$

(i) Find the Cartesian equation of the path the particle can move along. **1**

(ii) Sketch the graph of the path the particle can move along. **2**

Question 15 (15 marks) Start a new Writing Booklet.

a) Evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx$ in simplest exact form with a rational denominator. **4**

b) Let $I_n = \displaystyle\int_0^1 x^{2n}\sqrt{1-x^2}\,dx$ where $n = 0, 1, 2, ...$

(i) State the value of I_0. **1**

(ii) Use integration by parts to show that for $n = 1, 2, 3, ...$

$$I_n = \left(\frac{2n-1}{2n+2}\right) I_{n-1}$$ **3**

(iii) Hence calculate $\displaystyle\int_0^1 x^6\sqrt{1-x^2}\,dx$ **3**

c) The acceleration of a particle, moving in a straight line, is given by

$$\ddot{x} = (2x - 5) \quad \text{ms}^{-2}$$

where x is the particle's displacement.

The particle is initially at rest at $x = 6$.

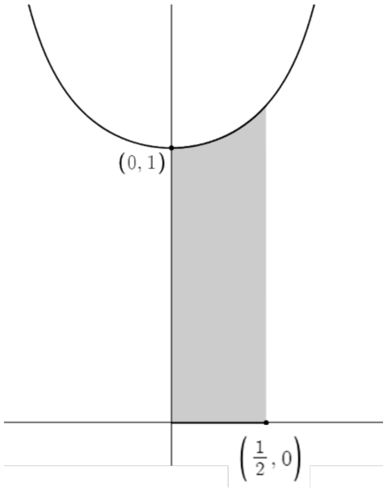
(i) Find an expression for v^2, where v is the particle's velocity. **2**

(ii) Does the particle return to a rest position? Justify your answer. **2**

Question 16 (15 marks) Start a new Writing Booklet.

a) The diagram shows a shaded region bounded by the graph of

$y = \dfrac{1}{\sqrt{1 - x^2}}$, the coordinate axes, and the line $x = \dfrac{1}{2}$.

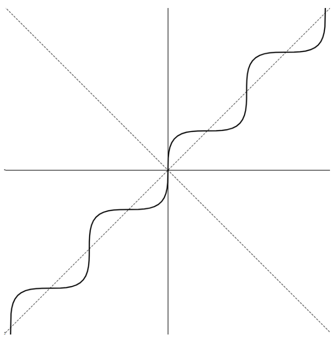


(i) Find the area of the region. **1**

(ii) Show that for $n \geq 2$,

$$\frac{1}{2} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^n}} \leq \frac{\pi}{6}$$ **3**

b) The graph below is obtained by rotating the graph of $y = \sin x$ anticlockwise about the origin by 45 degrees.



Show that the graph can be represented by the parametric equations:

$$x = \frac{t - \sin t}{\sqrt{2}}, \quad y = \frac{t + \sin t}{\sqrt{2}}$$

2

c)

(i) Prove for all $a, b > 0$ that $\dfrac{a+b}{2} \geq \sqrt{ab}$ **1**

(ii) Prove that if $x > 1$ then $\dfrac{x}{\sqrt{x-1}} \geq 2$ **2**

(iii) Hence or otherwise prove that for a , $b > 1$, the following inequality holds:

$$\frac{a^2}{b-1} + \frac{b^2}{a-1} \geq 8$$ **2**

d) Prove by mathematical induction that:

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x \; dx = \frac{2\pi(2n)!}{4^{n+1}(n!)^2} \text{ for all integers } n \geq 1$$ **4**

End of paper

2024 Mathematics Extension 2 Trial marking guidelines		
Q	**Solution**	**Marks**
1	*P* says "For every real number *x*, there is another real number *y* whose cube is *x*." This is true ($y = \sqrt[3]{x}$) *Q* says "There is a real number *y*, such that all real numbers *x* are equal to y^3 ." There is no such number, so this is false.	B
2	$\underset{\sim}{\mathbf{a}} \cdot (\underset{\sim}{\mathbf{a}} - 3\underset{\sim}{\mathbf{b}})$ $= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 9 \\ -3 \\ 6 \end{pmatrix} \right]$ $= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 5 \\ -3 \end{pmatrix}$ $= -8 + 10 - 9$ $= -7$	A
3	$0 < a < b$ $\therefore$ (a,b) is above the line $y = x$ and to the right of the y-axis $\therefore \quad \frac{\pi}{4} < z < \frac{\pi}{2}$ $\arg(z^4) = 4\arg z$ $\therefore \quad \pi < \arg(z^4) < 2\pi$ $\therefore$ z^4 is in Quadrant 3 or 4 $\therefore \quad Im(z^4) < 0$	C
4	$v^2 = 40 - 4x^2 + 32x$ $v^2 = 40 - 4(x^2 - 8x)$ $v^2 = 40 - 4\left[(x-4)^2 - 16\right]$ $v^2 = 40 - 4(x-4)^2 + 64$ $v^2 = 104 - 4(x-4)^2$ $v^2 = 4\left[26 - (x-4)^2\right]$ $v^2 = (2)^2\left[\left(\sqrt{26}\right)^2 - (x-4)^2\right]$ $n = 2$ $\therefore$ period $= \frac{2\pi}{2} = \pi$	C
5	$n^4 + (n+i)^4$ $= n^4 + n^4 + 4n^3i + 6n^2i^2 + 4ni^3 + i^4$ $= 2n^4 + 4n^3i - 6n^2 - 4ni + 1$ $= (2n^4 - 6n^2 + 1) + (4n^3 - 4n)i$ This is real when $4n^3 - 4n = 0$ $4n(n^2 - 1) = 0$ $n = 0, \pm 1$ And $(2n^4 - 6n^2 + 1)$ is an integer for these values $\therefore$ 3 values	C
6	If $f(x)$ and $g(x)$ are both odd, then	D

	$f(g(-x))$ $= f(-g(x))$ $= -f(g(x))$ Therefore $f(g(x))$ is odd. Therefore the original statement is true. Therefore the contrapositive is true. The converse is "If $f(g(x))$ is odd then $f(x)$ and $g(x)$ are both odd." This can be proven false by counterexample. e.g. Let $f(x) = x^2$ (not odd) and $g(x) = x^{1.5}$ (not odd) . Then $f(g(x)) = x^3$ (odd). So $f(g(x))$ can be odd with $f(x)$ or $g(x)$ not being odd.	
7	The Cartesian equations of the spheres are: $x^2 + y^2 + z^2 = 36$ ① and $(x-4)^2 + y^2 + z^2 = 9$ ② ② − ① $\rightarrow (x-4)^2 - x^2 = -27 \rightarrow -8x = -43 \rightarrow x = \frac{43}{8}$ $\therefore (D)$	(D)
8	$\omega^2 + \frac{1}{1+\omega^2} = \omega \rightarrow \omega^2(1+\omega)^2 = \omega(1+\omega)$ $\rightarrow \omega^4 - \omega^3 + \omega^2 - \omega + 1 = 0$ (from fifth roots of -1) $\rightarrow \omega^5 = -1$ $\rightarrow \omega^{2\,024} = \omega^4.\omega^{2\,020} = \omega^4.(\omega^5)^{404}$ $\rightarrow \omega^{2\,024} = \omega^4.(-1)^{404} = \omega^4$ $\therefore (D)$	(D)
9	$\lvert z\rvert^2 = \cos^4\theta + \sin^4\theta = (\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta$ $= 1 - \frac{1}{2}(\sin^2 2\theta)$ $0 \le \theta \le \frac{\pi}{2} \Rightarrow 0 \le \sin^2\theta \le 1 \Rightarrow 0 \le \frac{1}{2}\sin^2\theta \le \frac{1}{2}$ $\Rightarrow 0 \ge -\frac{1}{2}\sin^2\theta \ge -\frac{1}{2} \Rightarrow 1 \ge 1 - \frac{1}{2}\sin^2\theta \ge \frac{1}{2}$ $\Rightarrow \frac{1}{2} \le \lvert z\rvert^2 \le 1$ $\Rightarrow \frac{1}{\sqrt{2}} \le \lvert z\rvert \le 1$ $\therefore (B)$	(B)
10	(A) and (B) are odd functions so each integral is zero. (D) is an even function but is entirely **below** the x axis $\therefore$ Negative (C) is even function which is entirely above the x axis $\therefore$ (C)	(C)

11ai	$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $\overrightarrow{AB} = \begin{pmatrix}-1\\1\\2\end{pmatrix} - \begin{pmatrix}2\\4\\-3\end{pmatrix} = \begin{pmatrix}-3\\-3\\5\end{pmatrix}$ $\lvert\overrightarrow{AB}\rvert = \sqrt{(-3)^2 + (-3)^2 + 5^2} = \sqrt{43}$	1 mark correct answer
11aii	$\overrightarrow{OC} = m\underset{\sim}{\mathbf{i}} + 2\underset{\sim}{\mathbf{j}} - m^2\underset{\sim}{\mathbf{k}}$ $\overrightarrow{OA} \perp \overrightarrow{OC} \Rightarrow \overrightarrow{OA} \cdot \overrightarrow{OC} = 0$ $\begin{pmatrix}2\\4\\-3\end{pmatrix} \cdot \begin{pmatrix}m\\2\\-m^2\end{pmatrix} = 2m + 8 + 3m^2 = 0$ $3m^2 + 2m + 8 = 0$ $\Delta = 2^2 - 4 \times 3 \times 8 = -92 < 0$ $\therefore$ there is no m such that $\overrightarrow{OA} \perp \overrightarrow{OC}$	2 marks 1 mark for obtaining a correct quadratic equation in m; 1 mark for showing it has no real solutions
11b	$z_1 = 3 - i, \quad z_2 = 1 - 2i$ $\left\lvert \frac{z_1}{z_2} + k \right\rvert = \sqrt{k + 2}$ $\frac{z_1}{z_2} = \frac{3 - i}{1 - 2i} = 1 + i$ $\lvert 1 + i + k \rvert = \sqrt{k + 2}$ $\sqrt{(1 + k)^2 + 1^2} = \sqrt{k + 2}$ $k^2 + 2k + 2 = k + 2$ $k^2 + k = 0$ $k(k + 1) = 0$ $\therefore \quad k = 0,-1$	3 marks 1[st] mark for finding $\frac{z_1}{z_2}$, 2[nd] mark for using the modulus definition $\lvert x + iy \rvert = \sqrt{x^2 + y^2}$, 3[rd] mark for solving for k
11c	$z^2 - (1 - 2i)z = 7 + i \Rightarrow z^2 - (1 - 2i) - (7 + i) = 0$ $z = \frac{(1 - 2i) \pm \sqrt{(1 - 2i)^2 - 4.1. - (7 + i)}}{2}$ $z = \frac{(1 - 2i) \pm \sqrt{25}}{2} = \frac{(1 - 2i) \pm 5}{2}$ $\therefore \quad z_1 = \frac{6 - 2i}{2} \qquad z_2 = \frac{-4 - 2i}{2}$ $z_1 = 3 - i \qquad z_2 = -2 - i$	3 marks 1 mark for using quadratic formula, 1 mark for each solution

11d	$P(z) = z^4 - 2z^3 + az^2 + bz + 50$ $P(2 - i) = 0 \Rightarrow P(2 + i) = 0$ (since coefficients are real) $\therefore z^2 - 2Re(\alpha)z + \lvert\alpha\rvert^2 = 0$ is a quadratic factor $\Rightarrow z^2 - 2(2)z + 2^2 + (-1)^2 = z^2 - 4z + 5$ is a factor of $P(z)$ $z^4 - 2z^3 + az^2 + bz + 50 = (z^2 - 4z + 5)(z^2 + mz + n)$ $\therefore 50 = 5n \Rightarrow n = 10$ Term in z^3 : $-2z^3 = z^2.mz + (-4z.z^2) \Rightarrow -2z^3 = (m - 4)z^3$ $\therefore m - 4 = -2 \Rightarrow m = 2$ $\therefore P(z) = (z^2 - 4z + 5)(z^2 + 2z + 10)$	3 marks 2 marks for finding a or b correctly; or 2 marks for finding the zeroes of $P(z)$ without factorising it 1 mark for correctly substituting $z = 2 - i$ and simplifying, or equivalent merit
11e	$\sin x = \dfrac{2t}{1 + t^2} \quad \cos x = \dfrac{1 - t^2}{1 + t^2}$ $5 + 3\sin x - 4\cos x$ $= 5 + \dfrac{6t}{1 + t^2} - \dfrac{4(1 - t^2)}{1 + t^2}$ $= \dfrac{9t^2 + 6t + 1}{1 + t^2}$ $\dfrac{1}{5 + 3\sin x - 4\cos x}$ $= \dfrac{1 + t^2}{9t^2 + 6t + 1} = \dfrac{1 + t^2}{(3t + 1)^2}$ $t = \tan\left(\dfrac{x}{2}\right) \rightarrow dx = \dfrac{2}{1 + t^2}\, dt$ $x = \dfrac{\pi}{3} \rightarrow t = \tan\left(\dfrac{\pi}{6}\right) = \dfrac{1}{\sqrt{3}}$; $x = \dfrac{\pi}{2} \rightarrow t = \tan\left(\dfrac{\pi}{4}\right) = 1$	3 marks 1st mark for correctly finding the limits and dx in terms of t, or finding the integrand in terms of t 2nd mark for correctly finding the integral in terms of t 3rd mark for integrating and evaluating

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{5 + 3\sin x - 4\cos x} dx$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1 + t^2}{(3t + 1)^2} \times \frac{2}{1 + t^2} \, \mathrm{dt}$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} 2(3t + 1)^{-2} dt$$

$$= \left[-\frac{2}{3(3t + 1)} \right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$= -\frac{2}{3}\left[\frac{1}{4} - \frac{1}{3\left(\frac{1}{\sqrt{3}}\right) + 1}\right]$$

$$-\frac{2}{3}\left[\frac{1}{4} - \frac{1}{\sqrt{3} + 1}\right]$$

$$= -\frac{2}{3}\left(\frac{\sqrt{3} + 1 - 4}{4\left(\sqrt{3} + 1\right)}\right)$$

$$= -\left[\frac{\sqrt{3} - 3}{6\left(\sqrt{3} + 1\right)}\right]$$

$$= \frac{3 - \sqrt{3}}{6\left(\sqrt{3} + 1\right)} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{4\sqrt{3} - 6}{12}$$

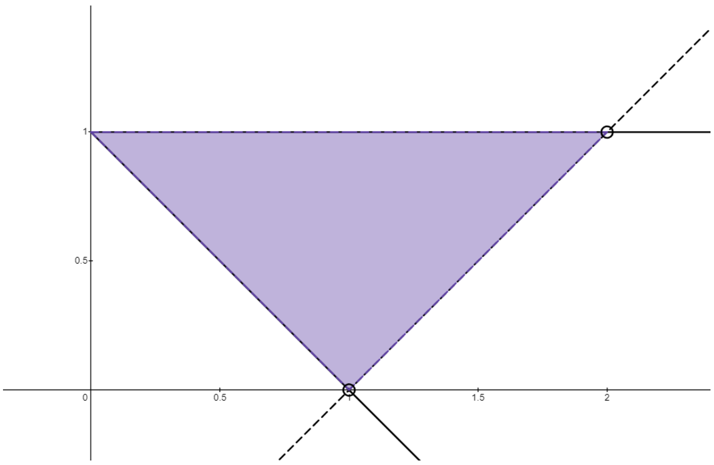
$$= \frac{2\sqrt{3} - 3}{6} \quad \text{or} \quad \frac{\sqrt{3}}{3} - \frac{1}{2} \quad \text{or} \quad \frac{1}{\sqrt{3}} - \frac{1}{2}$$

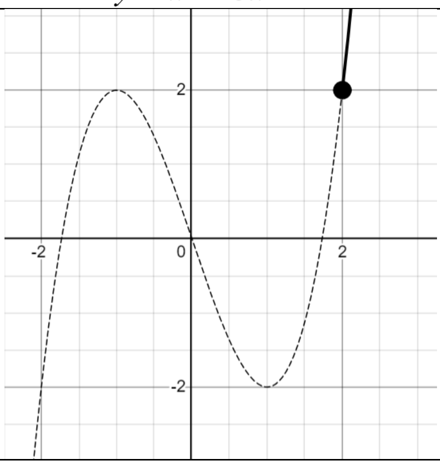
12a	$z = 2 + 2\sqrt{3}\ i$ $\lvert z\rvert = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = 4$ $Arg\ z = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$ $\therefore \quad z = 4\text{cis}\,\frac{\pi}{3}$ Let $w^4 = z$ $\Rightarrow \quad w_k = \sqrt[4]{4}\left[\text{cis}\left(\frac{\pi}{3} + 2k\pi\right)\right]^{\frac{1}{4}},\ k = 0,1,2,3$ $w_k = \sqrt{2}\ \text{cis}\left(\frac{\frac{\pi}{3} + 2k\pi}{4}\right),\ k = 0,1,2,3$ $w_k = \sqrt{2}\ \text{cis}\left(\frac{\pi}{12} + \frac{k\pi}{2}\right),\ k = 0,1,2,3$ $w_0 = \sqrt{2}\ \text{cis}\,\frac{\pi}{12},\ w_1 = \sqrt{2}\ \text{cis}\,\frac{7\pi}{12},$ $w_2 = \sqrt{2}\ \text{cis}\,\frac{13\pi}{12},\ w_3 = \sqrt{2}\ \text{cis}\,\frac{19\pi}{12}$	**2 Marks:** Correct answer **1 Mark:** Obtains one correct value
12bi	$\frac{3}{(x + 1)(x^2 + 5)} \equiv \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 5}$ $3 \equiv A(x^2 + 5) + (Bx + C)(x + 1)$ $3 \equiv (A + B)x^2 + (B + C)x + (5A + C)$ Equating coefficients: $A + B = 0 \Rightarrow A = -B$ $B + C = 0 \Rightarrow C = -B$ $5A + C = 3$ $\therefore \quad -5B - B = 3 \Rightarrow -6B = 3 \Rightarrow B = -\frac{1}{2}$ $\therefore \quad A = C = \frac{1}{2}$	**2 Marks:** Correct answer **1 Mark:** Obtains a correct value for *A, B* or *C*

12bii	$I = \int \left[\frac{1}{2} \frac{1}{x+1} + \frac{\frac{1}{2} - \frac{1}{2}x}{x^2+5} \right] dx$ $= \frac{1}{2} \int \left[\frac{1}{x+1} + \frac{1-x}{x^2+5} \right] dx$ $= \frac{1}{2} \left[\int \frac{1}{x+1}\, dx + \int \frac{1}{x^2+5}\, dx - \frac{1}{2} \int \frac{2x}{x^2+5}\, dx \right]$ $= \frac{1}{2} \left[\ln\lvert x+1 \rvert + \frac{1}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}} - \frac{1}{2} \ln\lvert x^2+5 \rvert \right] + C$	**2 Marks:** Correct answer **1 Mark:** Obtains one correct primitive function
12c	$u + v = 17i - j + 2k$ ① $u - v = i + 9j - 4k$ ② ① + ② : $2u = 18i + 8j - 2k$ $u = 9i + 4j - k$ ① − ② : $2v = 16i - 10j + 6k$ $v = 8i - 5j + 3k$ $u \cdot v = 9 \times 8 + 4 \times -5 - 1 \times 3 = 49$ $\lvert u \rvert = \sqrt{9^2 + 4^2 + (-1)^2} = \sqrt{98}$ $\lvert v \rvert = \sqrt{8^2 + (-5)^2 + 3^2} = \sqrt{98}$ $\cos\theta = \frac{u \cdot v}{\lvert u \rvert \lvert v \rvert} = \frac{49}{\sqrt{98} \times \sqrt{98}} = \frac{1}{2}$ $\theta = \cos^{-1} \frac{1}{2} = \frac{\pi}{3}$	**3 Marks:** Correct answer **2 Marks:** Correctly finds *u* and *v* and the scalar product of *u* and *v* **1 Mark:** Correctly finds one of *u* or *v* OR Correctly finds *u*•*v* from incorrect values of *u* and/or *v*
12d	Suppose there exist integers a and b such that $a^2 = 4b - 2$ Then, $a^2 = 2(2b-1)$ $\therefore a^2$ is even $\therefore a$ is even Let $a = 2k,\ k \in \mathbb{Z}$ $\therefore (2k)^2 = 4b - 2$ $4k^2 = 4b - 2$ $2k^2 = 2b - 1$ But LHS is even and RHS is odd − a contradiction $\therefore$ there are no integers a and b such that $a^2 = 4b - 2$	**3 Marks:** Correct answer **2 Marks:** Recognises that *a* is even and attempts to make further progress. **1 Mark:** Makes a correct statement of the contradiction and factorises $a^2 = 4b-2$

12e	The contrapositive statement is: "If **none** of *a,b,c* are divisible by 3 then $a^3 + b^3 + c^3$ is not divisible by 9". If each of *a, b* and *c* are not divisible by 3 then each one is of the form $3k + 1$ or $3k + 2$. Hence, a^3, b^3 and c^3 end in either 1 or 8. $\therefore a^3 + b^3 + c^3$ ends in either 3, 10, 17 or 24. $\rightarrow$ no common factor of 9 exists. $\therefore a^3 + b^3 + c^3$ is not divisible by 9. $\therefore$ The result is true by contraposition.	**3 Marks:** Correct answer **2 Marks:** States the correct contrapositive and attempts to form expressions involving remainders of 1and/or 2 **1 Mark:** States the correct contrapositive OR Attempts to use expressions of the form $3k$+1 or $3k$+2
13a	S(1): $7^1 + 13^1 + 19^1 = 39 = 13(3)$ Therefore S(1) is true Assume S(*k*): $7^k + 13^k + 19^k = 13M, \quad M \in \mathbb{Z}^+$, *k* odd RTP S(*k*+2) $7^{k+2} + 13^{k+2} + 19^{k+2}$ $= 13^{k+2} + 49(7^k) + 361(19^k)$ $= 13^{k+2} + 49[7^k + 19^k] + 312(19^k)$ $= 13^{k+2} + 49[13M - 13^k] + 312(19^k)$ from assumption $= 169(13^k) + 13(49M) - 49(13)^k + 13(24 \times 19^k)$ $= 120(13)^k + 13(49M) + 13(24 \times 19^k)$ As all terms are multiples of 13, $S(k + 2)$ holds Therefore $S(k) \Rightarrow S(k + 2)$ and the statement is true by induction	4 marks 1st mark for proving true for $n = 1$ 2nd mark for correctly writing $S(k)$ and $S(k + 2)$ (or equivalent) 3rd mark for using the assumption
13b	$\int_1^9 \frac{1}{\sqrt{x}\left(1 + \sqrt{x}\right)}\, dx$ Let $u^2 = x$. Then $dx = 2u$ du $x = 1 \rightarrow u = 1$ $x = 9 \rightarrow u = 3$ $I = \int_1^3 \frac{1}{u(1 + u)}\, 2u$ du $= \int_1^3 \frac{2}{1 + u}$ du $= 2[\ln(1 + u)]_1^3$ $= 2(\ln 4 - \ln 2)$ $= 2(2 \ln 2 - \ln 2)$ $= 2 \ln 2$	3 marks 1st mark for making a suitable substitution and for converting dx and the limits 2nd for correct integral in terms of u (which is easier to evaluate than the original integral) 3rd mark for integrating and evaluating

13ci	Let $z = x + iy$ $z = \overline{z} \Rightarrow x + iy = x - iy$ $\Rightarrow 2iy = 0$ $\Rightarrow y = 0$ $\Rightarrow z$ is real	1 mark
13cii	$\lvert z\rvert = \lvert w\rvert = 1$ $\therefore z\overline{z} = 1$ and $w\overline{w} = 1$ $\therefore \overline{z} = \frac{1}{z}$ and $\overline{w} = \frac{1}{w}$ Consider $\overline{\left(\frac{z+w}{1+zw}\right)}$ $\overline{\left(\frac{z+w}{1+zw}\right)} = \frac{\overline{z+w}}{\overline{(1+zw)}}$ $= \frac{\overline{z}+\overline{w}}{1+\overline{zw}}$ $= \frac{\overline{z}+\overline{w}}{1+\overline{z}\,\overline{w}}$ $= \frac{\frac{1}{z}+\frac{1}{w}}{1+\frac{1}{wz}}$ $= \frac{w+z}{wz} \div \frac{wz+1}{wz}$ $= \frac{z+w}{1+zw}$ Since $\frac{z+w}{1+zw}$ is equal to its own conjugate, it must be real (by part i)	3 marks 2 marks for considering the expression minus its conjugate and simplifying this 1 mark for demonstrating some understanding of conjugate properties, i.e. obtaining $\overline{\left(\frac{z+w}{1+zw}\right)} = \frac{\overline{z}+\overline{w}}{1+\overline{zw}}$, or equivalent
13d	$\lvert z-i\rvert < \lvert z-(2-i)\rvert$ is the set of all points which are nearer to (0, 1) than (2, -1). This is the region $y > x - 1$. $\frac{\pi}{4} \le \arg(iz+1) \le \frac{\pi}{2}$ $\frac{\pi}{4} \le \arg(i(z-i)) \le \frac{\pi}{2}$ $\frac{\pi}{4} \le \arg i + \arg(z-i) \le \frac{\pi}{2}$ $\frac{\pi}{4} \le \frac{\pi}{2} + \arg(z-i) \le \frac{\pi}{2}$ $-\frac{\pi}{4} \le \arg(z-i) \le 0$ This is the set of all complex numbers z such that the vector pointing from (0, 1) to z has argument between and $-\frac{\pi}{4}$ and 0.	4 marks 3 marks for the correct region but missing details such as open circles/ dotted lines/ solid lines 2 marks for sketching the line $y = x - 1$ as well as two rays 45 degrees apart in the wrong orientation/ position 1 mark for sketching the line $y = x - 1$

		
14ai	$l_1 : x = y = z$ $\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \end{pmatrix}$ $\underset{\sim}{\mathbf{r}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $l_2 : \underset{\sim}{\mathbf{r}} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ The lines are not parallel since $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ If the lines intersect, then there exists s and t real such that: $s\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\Rightarrow \quad s = t\,, \quad s = 3 - 2t, \quad s = t + 2$ Since $s = t$ contradicts $s = t + 2$, the lines do not intersect.	**3 Marks:** Correct answer **2 Marks:** Shows that l_1 and l_2 are non-parallel OR non- intersecting. **1 Mark:** Obtains a correct vector expression for l_1 and attempts to solve simultaneously with equation for l_2.
14aii	$\cos\theta = \dfrac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\sqrt{3}\,\sqrt{6}} = \dfrac{1 - 2 + 1}{3\sqrt{2}} = 0$ $\therefore \quad \theta = \dfrac{\pi}{2}$	**2 Marks:** Correct answer **1 Mark:** Finds an incorrect dot product and finds an angle consistent with that value.
14bi	$x = 1 + \sqrt{2}\,\cos 2t + \sqrt{2}\,\sin 2t$ $\dot{x} = -2\sqrt{2}\,\sin 2t + 2\sqrt{2}\,\cos 2t$ $\ddot{x} = -4\sqrt{2}\,\cos 2t - 4\sqrt{2}\,\sin 2t$ $\ddot{x} = -4\left[\sqrt{2}\,\cos 2t + \sqrt{2}\,\sin 2t\right]$ $\ddot{x} = -4(x - 1)$	**3 Marks:** Correct answer **2 Marks:** Correctly demonstrates motion is SH and finds one of the centre and period.

	Therefore the motion is simple harmonic with centre 1, $n = 2$ and period $\dfrac{2\pi}{2} = \pi$	**12 Mark:** One of three distinct parts is correct.
14bii	$x = 1 + \sqrt{2}\,(\cos 2t + \sin 2t)$ $R = \sqrt{1^2 + 1^2} = \sqrt{2}$ $\alpha = \tan^{-1}\left(\dfrac{1}{1}\right) = \dfrac{\pi}{4}$ $\therefore\ x = 1 + \sqrt{2} \times \sqrt{2}\cos\left(2t - \dfrac{\pi}{4}\right)$ $x = 1 + 2\cos\left(2t - \dfrac{\pi}{4}\right)$ $\therefore$ amplitude = 2	**2 Marks:** Correct answer **1 Mark:** Obtains a correct auxiliary angle expression.
14c	$y(y + 1) = x$ $y^2 + y = x$ $y^2 + y - x = 0$ $y = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-x)}}{2(1)}$ $y = \dfrac{-1 \pm \sqrt{1 + 4x}}{2}$ since $x \in \mathbb{R}^+$, $1 + 4x > 0$, $\therefore \sqrt{1 + 4x}$ exists and is real $\therefore$ y exists and is real	**2 Marks:** Correct answer **1 Mark:** Forms a correct discriminant expression.
14di	$r(t) = \begin{pmatrix} t + \dfrac{1}{t} \\ t^3 + \dfrac{1}{t^3} \end{pmatrix},\ t \in \mathbb{R}^+$ $x = t + \dfrac{1}{t},\quad y = t^3 + \dfrac{1}{t^3}$ $\left(t + \dfrac{1}{t}\right)^3 = t^3 + 3t^2\left(\dfrac{1}{t}\right) + 3t\left(\dfrac{1}{t^2}\right) + \dfrac{1}{t^3}$ $\left(t + \dfrac{1}{t}\right)^3 = \left(t^3 + \dfrac{1}{t^3}\right) + 3\left(t + \dfrac{1}{t}\right)$ $x^3 = y + 3x$ $y = x^3 - 3x$	**1 Mark:** Correct answer
14dii		**2 Marks:** Correct answer **1 Mark:** Draws a cubic curve for $x \geq 0$

	Since $t > 0$, $t + \frac{1}{t} \geq 2 \Rightarrow x \geq 2$ $(2)^3 - 3(2) = 2$, $\therefore$ closed circle at $(2, 2)$	
15a	$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1 + x^2}}\, dx$ Let $x = \tan\theta$. $dx = \sec^2\theta d\theta$ $x = \sqrt{3} \rightarrow \theta = \frac{\pi}{3}$ $x = 1 \rightarrow \theta = \frac{\pi}{4}$	4 marks 1st mark for making an appropriate substitution and for converting the limits and dx 2nd mark for obtaining a correct, simplified integral in terms of θ, i.e. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin^2\theta}\, d\theta$ or equivalent 3rd mark for finding the primitive 4th mark for evaluating and obtaining simplest form

	$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2\theta \sqrt{1 + \tan^2\theta}} \sec^2\theta \ d\theta$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2\theta \sqrt{\sec^2\theta}} \sec^2\theta \ d\theta$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2\theta \sec\theta} \sec^2\theta \ d\theta$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\tan^2\theta} \ d\theta$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos\theta} \div \frac{\sin^2\theta}{\cos^2\theta} \ d\theta$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin^2\theta} \ d\theta$ $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\cos\theta)(\sin\theta)^{-2} \ d\theta$ $= \left[-(\sin\theta)^{-1} \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $= -\left[\frac{1}{\sin\frac{\pi}{3}} - \frac{1}{\sin\frac{\pi}{4}} \right]$ $= -\left(\frac{2}{\sqrt{3}} - \sqrt{2} \right)$ $= -\frac{2}{\sqrt{3}} + \sqrt{2}$ $= -\frac{2\sqrt{3}}{3} + \sqrt{2}$ $= \frac{3\sqrt{2} - 2\sqrt{3}}{3}$	
15bi	$I_n = \int_0^1 x^{2n}\sqrt{1 - x^2} \ dx$ $I_0 = \int_0^1 x^0\sqrt{1 - x^2} \ dx$ $= \int_0^1 \sqrt{1 - x^2} \ dx$ $= \frac{1}{4} \pi(1)^2 = \frac{\pi}{4}$ (area of a quadrant)	1 mark

15bii	$I_n = \int_0^1 x^{2n}\sqrt{1-x^2}\;dx$ $= \int_0^1 x^{2n-1}\,.\,x\sqrt{1-x^2}\,dx$ Integration by parts: Let $u = x^{2n-1}$ $\quad dv = x(1-x^2)^{\frac{1}{2}}$ $du = (2n-1)x^{2n-2}$ $\quad v = -\frac{1}{3}(1-x^2)^{\frac{3}{2}}$ $I_n = \left[-\frac{1}{3}x^{2n-1}(1-x^2)^{\frac{3}{2}}\right]_0^1 - \int_0^1 -\frac{1}{3}(1-x^2)^{\frac{3}{2}}\,.\,(2n-1)\,x^{2n-2}$ $I_n = 0 + \frac{2n-1}{3}\int_0^1 (1-x^2)\sqrt{1-x^2}\;x^{2n-2}\,dx$ $I_n = \frac{2n-1}{3}\int_0^1 \left[x^{2n-2}\sqrt{1-x^2} - x^2.\,x^{2n-2}\sqrt{1-x^2}\,\right]dx$ $I_n = \frac{2n-1}{3}\left[\int_0^1 x^{2(n-1)}\sqrt{1-x^2}\;dx - \int_0^1 x^{2n}\sqrt{1-x^2}\;dx\right]$ $I_n = \frac{2n-1}{3}\,[I_{n-1} - I_n]$ $3I_n = (2n-1)I_{n-1} - (2n-1)I_n$ $I_n[3 + (2n-1)] = (2n-1)I_{n-1}$ $I_n(2n+2) = (2n-1)I_{n-1}$ $I_n = \frac{2n-1}{2n+2}\,I_{n-1}$	3 marks 2 marks for correctly using IBP with the correct choice of *u* and *v*', and attempting to simplify the integral 1 mark for some attempt at using integration by parts e.g. using IBP correctly with the wrong choice of *u* and *v*')
15biii	$\int_0^1 x^6\sqrt{1-x^2}\;dx$ $= I_3$ $= \frac{5}{8}\,I_2$ $= \frac{5}{8}\times\frac{3}{6}\,I_1$ $= \frac{5}{8}\times\frac{3}{6}\times\frac{1}{4}\,I_0$ $= \frac{5}{8}\times\frac{3}{6}\times\frac{1}{4}\times\frac{\pi}{4}$ $= \frac{5\pi}{256}$	3 marks 2 marks for correct solution with one error (e.g. mistaking the integral for I_6) 1 mark for one correct application of the reduction formula

15ci	$\ddot{x} = 2x - 5$ $v\dfrac{dv}{dx} = 2x - 5$ $\int v\,dv = \int (2x-5)\,dx$ $\dfrac{v^2}{2} = x^2 - 5x + C$ At $x = 6$, $v = 0$ $\Rightarrow \quad 0 = 36 - 30 + C \Rightarrow C = -6$ $\dfrac{v^2}{2} = x^2 - 5x - 6$ $v^2 = 2x^2 - 10x - 12$	2 marks 1[st] mark for using $v\dfrac{dv}{dx}$ or $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$ 2[nd] mark for integrating and using the initial condition
15cii	$v^2 = 2(x-6)(x+1)$ Particle can only be at rest when $x = -1$ or 6 Initially, the particle is at rest at $x = 6$ with positive acceleration $\ddot{x} = 7\ ms^{-2}$, which means it must then move to the right. Since for all $x \geq 6$, $\ddot{x} > 0$, the particle will always keep speeding up to the right, never again reaching $x = -1$ or 6. Hence it will never return to a rest position. Another explanation is: Velocity is initially 0. Acceleration being positive means Velocity is increasing. Since Acceleration is always positive, the Velocity is always increasing. Hence Velocity will not come back to 0.	2 marks for correct conclusion and explanation 1 mark for correct conclusion with incomplete explanation / partially incorrect explanation (0 marks for correct conclusion with incorrect reasons)
16ai	$A = \int_0^{\frac{1}{2}} \dfrac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1} x\right]_0^{\frac{1}{2}}$ $= \left[\sin^{-1}\dfrac{1}{2} - \sin^{1} 0\right]$ $= \dfrac{\pi}{6} - 0$ $= \dfrac{\pi}{6}$	**1 Mark:** Correct answer
16aii	For $n \geq 2$,	**3 Marks:** Correct answer **2 Marks:** Shows that $1 - x^n \geq 1 - x^2$ and establishes one side of the inequality. **1 Mark:** Explains why $0 \leq x \leq ½ \rightarrow 0 \leq x^n \leq x^2$

	$0 \le x \le \frac{1}{2}$ $\Rightarrow \quad 0 \le x^n \le x^2$ $\Rightarrow \quad 0 \ge -x^n \ge -x^2$ $\Rightarrow \quad 1 \ge 1 - x^n \ge 1 - x^2$ $\Rightarrow \quad 1 \ge \sqrt{1 - x^n} \ge \sqrt{1 - x^2}$ $\Rightarrow \quad 1 \le \frac{1}{\sqrt{1 - x^n}} \le \frac{1}{\sqrt{1 - x^2}}$ $\Rightarrow \int_0^{\frac{1}{2}} 1\, dx \le \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1 - x^n}}\, dx \le \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1 - x^2}}\, dx$ $[x]_0^{\frac{1}{2}} \le \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1 - x^n}}\, dx \le \frac{\pi}{6}$ from (i) $\frac{1}{2} \le \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1 - x^n}}\, dx \le \frac{\pi}{6}$	
16b	A point on the curve $y = \sin x$ is of the form $(x, \sin x)$, which can be represented by the complex number $z = x + i\sin x$. Rotating anticlockwise by 45 degrees about O is obtained by multiplying by $\text{cis}\frac{\pi}{4}$: $z' = \text{cis}\frac{\pi}{4}(x + i\sin x)$ $= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)(x + i\sin x)$ $= \frac{x}{\sqrt{2}} + \frac{x}{\sqrt{2}}i + \frac{\sin x}{\sqrt{2}}i - \frac{\sin x}{\sqrt{2}}$ $= \left(\frac{x - \sin x}{\sqrt{2}}\right) + i\left(\frac{x + \sin x}{\sqrt{2}}\right)$ Hence the new coordinates are $\left(\frac{x - \sin x}{\sqrt{2}}, \frac{x + \sin x}{\sqrt{2}}\right)$ Hence the parametric equations are $x = \frac{t - \sin t}{\sqrt{2}}$, $y = \frac{t + \sin t}{\sqrt{2}}$	**2 Marks:** Correct answer **1 Mark:** Attempts to rotate $y = \sin x$ by multiplying by cis(π/4)
16ci	$\left(\sqrt{a} - \sqrt{b}\right)^2 \ge 0$ $a - 2\sqrt{ab} + b \ge 0$ $a + b \ge 2\sqrt{ab}$ $\frac{a + b}{2} \ge \sqrt{ab}$	**1 Mark:** Correct answer
16cii	Method 1:	**2 Marks:** Correct answer

	$$\left(\sqrt{x-1}-1\right)^2 \ge 0$$ $$x-1-2\sqrt{x-1}+1 \ge 0$$ $$x-2\sqrt{x-1} \ge 0$$ $$x \ge 2\sqrt{x-1}$$ $$\frac{x}{\sqrt{x-1}} \ge 2$$ Method 2: $$\frac{x}{\sqrt{x-1}} = \frac{x-1}{\sqrt{x-1}} + \frac{1}{\sqrt{x-1}}$$ $$= \sqrt{x-1} + \frac{1}{\sqrt{x-1}}$$ $$\ge 2\sqrt{\sqrt{x-1} \times \frac{1}{\sqrt{x-1}}}$$ $$= 2$$	**1 Mark:** Forms an expression equivalent to $((\sqrt{x}-1)-1)^2$
16ciii	From (i), $$\frac{a^2}{b-1} + \frac{b^2}{a-1} \ge 2\sqrt{\frac{a^2}{b-1} \times \frac{b^2}{a-1}}$$ $$= 2\sqrt{\frac{a^2}{b-1}}\sqrt{\frac{b^2}{a-1}}$$ $$= 2 \times \frac{a}{\sqrt{b-1}} \times \frac{b}{\sqrt{a-1}}$$ $$\ge 2 \times 2 \times 2$$ $$= 8$$	**2 Marks:** Correct answer **1 Mark:** Obtains the result $\frac{a^2}{\boldsymbol{a-1}} + \frac{\boldsymbol{b^2}}{\boldsymbol{b-1}} \ge \mathbf{8}$
16d	$$\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \frac{2\pi(2n)!}{4^{n+1}(1!)^2}$$ S(1): $$\text{L}HS = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$$ $$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$ $$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$ $$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\sin\pi\right] - \frac{1}{2}(0-0)$$ $$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right]$$ $$= \frac{\pi}{4} = RHS$$	**4 Marks:** Correct answer **3 Marks:** Shows S(1) is true, makes correct statements for S(k) and S(k+1) and relevant progress toward showing S(k) → S(k+1) **2 Marks:** Shows S(1) is true and correctly states S(k) and S(k+1) **1 Mark:** Shows S(1) is true OR correctly states S(k) and S(k+1).

	Therefore S(1) is true. Assume S(k) is true: $\int_0^{\frac{\pi}{2}} \sin^{2k}x \; dx = \frac{2\pi(2k)!}{4^{k+1}(k!)^2}$ RTP S($k+1$): $\int_0^{\frac{\pi}{2}} \sin^{2(k+1)}x \; dx = \frac{2\pi[2(k+1)]!}{4^{k+2}[(k+1)!]^2}$ $LHS = \int_0^{\frac{\pi}{2}} \sin^{2k+2} x \; dx = \int_0^{\frac{\pi}{2}} \sin^{2k}x \sin^2 x \; dx$ $= \int_0^{\frac{\pi}{2}} \sin^{2k}x \; (1-\cos^2 x) \; dx$ $= \int_0^{\frac{\pi}{2}} \sin^{2k}x \; dx - \int_0^{\frac{\pi}{2}} \sin^{2k}x \; \cos^2 x \; dx$ Using integration by parts for $\int_0^{\frac{\pi}{2}} \sin^{2k}x \; \cos^2 x \; dx = \int_0^{\frac{\pi}{2}} \cos x .[\cos x \; \sin^{2k}x \;] \; dx$ Let $u = \cos x \qquad dv = \cos x(\sin x)^{2k}$ $du = -\sin x \qquad v = \frac{\sin^{2k+1}x}{2k+1}$	

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{2k}x \ \cos^2 x dx$$

$$= \left[\frac{\sin^{2k+1}x \cos x}{2k+1}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{-\sin^{2k+1}x \sin x}{2k+1} \, dx$$

$$= [0 - 0] + \int_0^{\frac{\pi}{2}} \frac{\sin^{2k+2}x}{2k+1} \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^{2k+2}x}{2k+1} \, dx$$

$$\therefore \qquad \int_0^{\frac{\pi}{2}} \sin^{2(k+1)}x \ \ dx = \int_0^{\frac{\pi}{2}} \sin^{2k} x \ \ dx - \int_0^{\frac{\pi}{2}} \frac{\sin}{2}$$

$$\left[1 + \frac{1}{2k+1}\right]\int_0^{\frac{\pi}{2}} \sin^{2k+2}x \ \ dx = \int_0^{\frac{\pi}{2}} \sin^{2k} x \ \ dx$$

$$\frac{2k+2}{2k+1}\int_0^{\frac{\pi}{2}} \sin^{2k+2}x \ \ dx = \frac{2\pi(2k)!}{4^{k+1}(k!)^2} \ \ \text{(from assum}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2k+2} x \ \ dx = \frac{2\pi(2k+1)(2k)!}{4^{k+1} \ (2k+2)(k!)^2}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2k+2}x \ \ dx = \frac{2\pi(2k+2)!}{4^{k+1} \ (2k+2)^2(k!)^2}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2k+2}x \ \ dx = \frac{2\pi[2(k+1)]!}{4^{k+1} \left([2(k+1)](k!)\right)^2}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2k+2}x \ \ dx = \frac{2\pi[2(k+1)]!}{4^{k+1} \ (4)[(k+1)k!]^2}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2k+2}x \ \ dx = \frac{2\pi[2(k+1)]!}{4^{k+2}[(k+1)!]^2}$$

Therefore S(k) => S(k + 1), therefore the statement is true by induction.